SINGLE POINT ADJUSTMENT WITHIN EXISTING NETWORKS BY MEANS OF THE repro-BLE

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Key words: Gauss-Markov model; Best Linear Estimate (BLE); Empirical BLE; repro-BLE

ABSTRACT

It is well known that, within a Gauss-Markov Model (GMM), the Best Linear Estimate (BLE) shows smaller Mean Squared Errors (MSE) for the estimated parameters if compared with the more traditional BLUUE (Best Linear Uniformly Unbiased Estimate). This result, however, is only of theoretical value as the BLE itself cannot be numerically evaluated without further assumptions. One common approach is based on the use of the BLUUE whenever an approximation of the BLE is required which leads to the so-called "empirical BLE."

In contrast, Schaffrin (2000) had argued that, at least in the univariate case of "direct/replicated" observations of one parameter, the so-called "reproducing BLE" should be superior to the empirical BLE whenever it exists. Both are, of course, nonlinear estimates in the end, and the formulae for their MSE can only be approximations.

Here, we shall consider the 2-D case where we put the necessary generalizations of the repro-BLE to work as presented by Schaffrin and Xu (2017). Concretely, a single point will be adjusted within an (existing) planar geodetic network for which we compare the BLUUE, the empirical BLE, and the repro-BLE (after checking the existence condition) of the point's coordinates.

INTRODUCTION

The reproBLE had first been introduced by Schaffrin (2000) for the 1-D case of direct/replicated observations. There, it had been argued that the reproBLE should, in general, turn out superior to the more commonly used empirical BLE whenever it exists. Now, we want to extend our discussion to the 2-D case where the coordinates of a single new point are to be determined in relation to an existing planar geodetic network.

Based on the initial work by Schaffrin and Xu (2017), we shall develop the "locus" of all estimates of type reproBLE (an ellipse), should they exist at all. Out of these infinitely many estimates, we shall identify four that seem to be of major interest, due to their particular geodetic properties, and compare them with the more traditional estimates of type BLUUE and empirical BLE.

1. THE BIVARIATE GAUSS-MARKOV MODEL

1.1. BLUUE and BLE

Let the model be defined by

$$\boldsymbol{y} = \underset{n\times 2}{A}\boldsymbol{\xi} + \boldsymbol{e}, \;\; \boldsymbol{e} \sim (\underset{n\times 1}{\boldsymbol{0}}, \underset{n\times n}{\Sigma} = \sigma_0^2 P^{-1}), \quad (1)$$

where, after linearization, A is the $n \times 2$ coefficient matrix with rk $A = 2$, and $\boldsymbol{\xi}$ is the unknown 2×1 parameter vector. Moreover, $\boldsymbol{y}$ is the $n \times 1$ vector of observational increments, $\boldsymbol{e}$ is the (unknown) $n \times 1$ random error vector whose expectation is $E\{\boldsymbol{e}\} = \boldsymbol{0}$, while its dispersion matrix $D\{\boldsymbol{e}\} = \sigma_0^2 P^{-1}$ is split into the product of an (unknown) variance component σ_0^2 and a (given) $n \times n$ cofactor matrix P^{-1} (whose inverse is the weight matrix P).

It is well known that, in model (1), the LEast Squares Solution (LESS) represents the BLUUE (Best Linear Uniformly Unbiased Estimate) and is given by

$$\hat{\boldsymbol{\xi}} = N^{-1}\boldsymbol{c} \sim (\boldsymbol{\xi}, \sigma_0^2 N^{-1}) \;\text{ for }\; [N, \boldsymbol{c}] = A^T P[A, \boldsymbol{y}], \quad (2)$$

meaning that the expectation $E\{\hat{\boldsymbol{\xi}}\} = \boldsymbol{\xi}$ for all $\boldsymbol{\xi} \in \mathbb{R}^m$ and that the dispersion matrix

$$D\{\hat{\boldsymbol{\xi}}\} = \sigma_0^2 N^{-1} = MSE\{\hat{\boldsymbol{\xi}}\}, \quad (3)$$

therefore, equals the Mean Squared Error matrix. In addition, an unbiased estimate of σ_0^2 can be obtained via

$$\hat{\sigma}_0^2 = (n-2)^{-1}(\boldsymbol{y}^T P \boldsymbol{y} - \boldsymbol{c}^T \hat{\boldsymbol{\xi}}) \sim \\ \sim (\sigma_0^2, 2(\sigma_0^2)^2/(n-2)). \quad (4)$$

In contrast, the Best Linear Estimate (BLE) of $\boldsymbol{\xi}$ is derived from the principle

$$\text{tr}\, MSE\{\bar{\boldsymbol{\xi}} = L\boldsymbol{y}\} = \text{tr}[D\{\bar{\boldsymbol{\xi}} = L\boldsymbol{y}\} + \\ + E\{L\boldsymbol{y} - \boldsymbol{\xi}\} \cdot E\{L\boldsymbol{y} - \boldsymbol{\xi}\}^T] = \sigma_0^2 \cdot [\text{tr}(LP^{-1}L^T) + \\ \text{tr}(LA - I_m)\boldsymbol{\xi}\sigma_0^{-2}\boldsymbol{\xi}^T(LA - I_m)^T] = \min_{L^T}. \quad (5)$$

The necessary condition for the BLE $\bar{\boldsymbol{\xi}} = \underset{2\times n}{L}\boldsymbol{y}$ then reads

$$\frac{1}{2}\frac{\partial trMSE\{\bar{\boldsymbol{\xi}}\}}{\partial L^T} =$$

$$= (P^{-1} + A\boldsymbol{\xi}\sigma_0^{-2}\boldsymbol{\xi}^T A^T)\cdot \bar{L}^T - A\boldsymbol{\xi}\sigma_0^{-2}\boldsymbol{\xi}^T \doteq 0, \tag{6a}$$

respectively

$$\begin{aligned}\bar{L} &= \boldsymbol{\xi}\sigma_0^{-2}\boldsymbol{\xi}^T A^T \cdot (I_n + PA\boldsymbol{\xi}\sigma_0^{-2}\boldsymbol{\xi}^T A^T)^{-1}P = \\ &= \boldsymbol{\xi}\cdot\sigma_0^{-2}(1+\boldsymbol{\xi}^T N\boldsymbol{\xi}\sigma_0^{-2})^{-1}\cdot\boldsymbol{\xi}^T A^T P, \end{aligned} \tag{6b}$$

leading to the representation

$$\bar{\boldsymbol{\xi}} = \bar{L}\boldsymbol{y} = \boldsymbol{\xi}\cdot(\bar{\boldsymbol{\xi}}^T\boldsymbol{c})/(\sigma_0^2 + \boldsymbol{\xi}^T N\boldsymbol{\xi}). \tag{7}$$

The bias of the BLE can now be computed as

$$E\{\bar{\boldsymbol{\xi}}\} - \boldsymbol{\xi} = -\boldsymbol{\xi}\cdot\sigma_0^2/(\sigma_0^2 + \boldsymbol{\xi}^T N\boldsymbol{\xi}), \tag{8a}$$

and its Mean Squared Error matrix as

$$\begin{aligned} MSE\{\bar{\boldsymbol{\xi}}\} = \sigma_0^2 \cdot [&\boldsymbol{\xi}(\sigma_0^2 + \boldsymbol{\xi}^T N\boldsymbol{\xi})^{-1}\boldsymbol{\xi}^T N\boldsymbol{\xi} \\ &(\sigma_0^2 + \boldsymbol{\xi}^T N\boldsymbol{\xi})^{-1}\boldsymbol{\xi}^T + \boldsymbol{\xi}(\sigma_0^2 + \boldsymbol{\xi}^T N\boldsymbol{\xi})^{-1}\cdot \\ &\cdot\sigma_0^2\cdot\sigma_0^{-2}\cdot\sigma_0^2(\sigma_0^2 + \boldsymbol{\xi}^T N\boldsymbol{\xi})^{-1}\boldsymbol{\xi}^T], \end{aligned}$$

or

$$MSE\{\bar{\boldsymbol{\xi}}\} = \sigma_0^2\cdot\boldsymbol{\xi}(\sigma_0^2 + \boldsymbol{\xi}^T N\boldsymbol{\xi})^{-1}\boldsymbol{\xi}^T, \tag{8b}$$

which is a *rank-1* matrix.

1.2. The empirical BLE and the reproBLE

Since the BLE $\bar{\boldsymbol{\xi}}$ from (7) cannot be evaluated without further knowledge of $\boldsymbol{\xi}$ itself, it has been proposed to replace $\boldsymbol{\xi}$ by the BLUUE $\hat{\boldsymbol{\xi}}$, thus leading to the empirical BLE

$$\begin{aligned}\hat{\bar{\boldsymbol{\xi}}} &= \hat{\boldsymbol{\xi}}\cdot(\hat{\sigma}_0^2 + \hat{\boldsymbol{\xi}}^T N\hat{\boldsymbol{\xi}})^{-1}(\hat{\boldsymbol{\xi}}^T\boldsymbol{c}) = \\ &= \hat{\boldsymbol{\xi}}\cdot\frac{\boldsymbol{c}^T N^{-1}\boldsymbol{c}}{\hat{\sigma}_0^2 + \boldsymbol{c}^T N^{-1}\boldsymbol{c}} = \qquad (9a) \\ &= \hat{\boldsymbol{\xi}}\left[\frac{(n-2)(\boldsymbol{c}^T N^{-1}\boldsymbol{c})}{\boldsymbol{y}^T P\boldsymbol{y} + (n-3)(\boldsymbol{c}^T N^{-1}\boldsymbol{c})}\right], \qquad (9b)\end{aligned}$$

which turns out proportional to the BLUUE $\hat{\boldsymbol{\xi}}$, but somewhat shorter than $\hat{\boldsymbol{\xi}}$, hence also known as "shrinkage estimate;" cf. Gruber (1998). For other choices, see Xu (1998).

Its Mean Squared Error matrix may now be approximated by replacing $\boldsymbol{\xi}$ and σ_0^2 in (8b) by $\hat{\boldsymbol{\xi}}$ and $\hat{\sigma}_0^2$, leading to

$$MSE\{\hat{\bar{\boldsymbol{\xi}}}\} \approx \sigma_0^2\cdot\hat{\boldsymbol{\xi}}(\hat{\sigma}_0^2 + \boldsymbol{c}^T N^{-1}\boldsymbol{c})^{-1}\hat{\boldsymbol{\xi}}^T \tag{10}$$

which, again, is a *rank-1* matrix.

In contrast, for the reproducing BLE (reproBLE), the estimate to replace $\boldsymbol{\xi}$ and the right side of (7) ought to be the same as the left side. Thus, we have to solve the nonlinear equation

$$\bar{\bar{\boldsymbol{\xi}}} = \bar{\bar{\boldsymbol{\xi}}}(\sigma_0^2 + \bar{\bar{\boldsymbol{\xi}}}^T N\bar{\bar{\boldsymbol{\xi}}})^{-1}(\bar{\bar{\boldsymbol{\xi}}}^T\boldsymbol{c}), \tag{11a}$$

or, after excluding the trivial solution ($\bar{\bar{\boldsymbol{\xi}}} = \boldsymbol{0}$),

$$\sigma_0^2 + \bar{\bar{\boldsymbol{\xi}}}^T N\bar{\bar{\boldsymbol{\xi}}} - \bar{\bar{\boldsymbol{\xi}}}^T N\hat{\boldsymbol{\xi}} = 0, \tag{11b}$$

respectively

$$\left(\bar{\bar{\boldsymbol{\xi}}} - \frac{1}{2}\hat{\boldsymbol{\xi}}\right)^T N\left(\bar{\bar{\boldsymbol{\xi}}} - \frac{1}{2}\hat{\boldsymbol{\xi}}\right) = \frac{1}{4}\cdot\hat{\boldsymbol{\xi}}^T N\hat{\boldsymbol{\xi}} - \sigma_0^2, \tag{11c}$$

which represents an ellipse as location for all estimates of type reproBLE as long as the inequality

$$\boxed{4\sigma_0^2 < \boldsymbol{\xi}^T N\boldsymbol{\xi}} \tag{12}$$

is guaranteed. Obviously, when σ_0^2 goes to 0, the BLUUE $\hat{\boldsymbol{\xi}}$ itself becomes one of the reproBLEs. Vice versa, when σ_0^2 approaches $\hat{\boldsymbol{\xi}}^T N\hat{\boldsymbol{\xi}}/4$, one half of the BLUUE becomes the only existing reproBLE (and the ellipse shrinks to this point).

2. reproBLEs of interest

Following up on the above observation, for reasonably small values of σ_0^2, the interesting choices of the reproBLE should all be smaller than the BLUUE (shrinkage estimates), but would fall in its neighborhood. We identified four of them on the basis of their geometric properties, each of which are depicted in Figure 1.

- $\bar{\bar{\boldsymbol{\xi}}}^1$ is the endpoint of the major ellipsoidal axis;
- $\bar{\bar{\boldsymbol{\xi}}}^2$ is the farthest point of the ellipse from the origin;
- $\bar{\bar{\boldsymbol{\xi}}}^3$ is the orthogonal projection of $\hat{\boldsymbol{\xi}}$ onto the ellipse;
- $\bar{\bar{\boldsymbol{\xi}}}^4$ is the far ellipse point that is proportional to $\hat{\boldsymbol{\xi}}$.

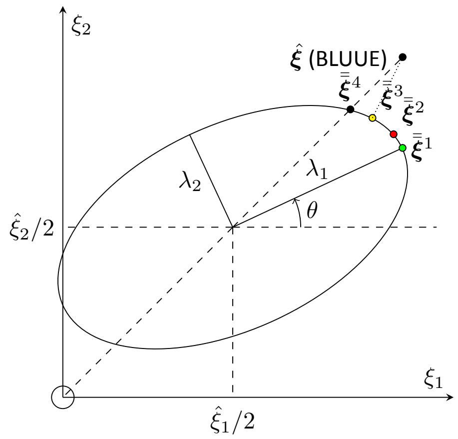

Figure 1: Ellipse centered at $(1/2)\hat{\boldsymbol{\xi}}$, with rotation angle θ, semi-major axis length λ_1, and semi-minor axis length λ_2

Let us now derive the corresponding formulas for the above four points of interest.

Point 1: First, we scale the matrix N to arrive at the new matrix

$$N_{\text{scaled}} := (\frac{1}{4}\cdot\hat{\boldsymbol{\xi}}^T N\hat{\boldsymbol{\xi}} - \sigma_0^2)^{-1}\cdot N = U\cdot\Lambda^2\cdot U^T \tag{13a}$$

which we decompose into eigenvalues and eigenvectors, with

$$U := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ and } \Lambda := \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. \tag{13b}$$

Then we can write

$$\boxed{\begin{aligned} \bar{\bar{\xi}}_1^1 &= \frac{1}{2}\cdot\hat{\xi}_1 + \lambda_1 \cdot \cos\theta, \quad (14a) \\ \bar{\bar{\xi}}_2^1 &= \frac{1}{2}\cdot\hat{\xi}_2 + \lambda_2 \cdot \sin\theta. \quad (14b) \end{aligned}}$$

Point 2: Secondly, we maximize $\bar{\bar{\xi}}^T\bar{\bar{\xi}}$ subject to (11b), resp. make the Lagrange target function

$$\Phi(\bar{\bar{\xi}},\lambda) := \bar{\bar{\xi}}^T\bar{\bar{\xi}} + \lambda\cdot(\sigma_0^2 + \bar{\bar{\xi}}^T N\bar{\bar{\xi}} - \bar{\bar{\xi}}^T N\hat{\xi}) \tag{15}$$

stationary. This leads to the necessary condition

$$\frac{1}{2}\frac{\partial\Phi}{\partial\bar{\bar{\xi}}} = \bar{\bar{\xi}}^2 + N\bar{\bar{\xi}}^2\cdot\hat{\lambda} - \frac{1}{2}\boldsymbol{c}\cdot\hat{\lambda} \doteq \boldsymbol{0} \tag{16a}$$

and consequently to

$$\boxed{\begin{aligned} \bar{\bar{\xi}}^2 &= \frac{1}{2}(I_2 + \hat{\lambda}\cdot N)^{-1}\boldsymbol{c}\cdot\lambda = \\ &= \frac{1}{2}(\hat{\lambda}^{-1}\cdot I_2 + N)^{-1}\boldsymbol{c}. \end{aligned}} \tag{16b}$$

The other necessary condition will then come from

$$\frac{\partial\Phi}{\partial\lambda} = \sigma_0^2 + (\bar{\bar{\xi}}^2)^T N\bar{\bar{\xi}}^2 - (\bar{\bar{\xi}}^2)^T\boldsymbol{c} \doteq 0, \tag{17a}$$

which, after inserting (16b), leads to the polynomial

$$\begin{aligned} 4\sigma_0^2 + \boldsymbol{c}^T(\hat{\lambda}^{-1}\cdot I_2 + N)^{-1}N(\hat{\lambda}^{-1}\cdot I_2 + N)^{-1}\boldsymbol{c} - \\ - 2\boldsymbol{c}^T(\hat{\lambda}^{-1}\cdot I_2 + N)^{-1}\boldsymbol{c} = 0, \end{aligned} \tag{17b}$$

or

$$\boxed{\begin{aligned} 4\sigma_0^2 + \boldsymbol{c}^T(\hat{\lambda}^{-1}\cdot I_2 + N)^{-1}\boldsymbol{c} &= \\ = \hat{\lambda}^{-1}\cdot\boldsymbol{c}^T(\hat{\lambda}^{-1}\cdot I_2 + N)^{-2}\boldsymbol{c}, \end{aligned}} \tag{17c}$$

which needs to be solved for $\hat{\lambda}^{-1}$. The inverse Lagrange multiplier can then be re-implemented into (16b) to find $\bar{\bar{\xi}}^2$.

Point 3: Thirdly, we minimize the distance between the BLUUE $\hat{\xi}$ and $\bar{\bar{\xi}}^3$, namely $(\bar{\bar{\xi}}^3 - \hat{\xi})^T(\bar{\bar{\xi}}^3 - \hat{\xi})$, subject to (11b), resp. make the Lagrange target function

$$\Phi(\bar{\bar{\xi}},\lambda) := \bar{\bar{\xi}}^T\bar{\bar{\xi}} - 2\bar{\bar{\xi}}^T\hat{\xi} + \hat{\xi}^T\hat{\xi} + \lambda\cdot[\sigma_0^2 + \bar{\bar{\xi}}^T N(\bar{\bar{\xi}} - \hat{\xi})] \tag{18}$$

stationary. This leads to the necessary condition

$$\frac{1}{2}\frac{\partial\Phi}{\partial\bar{\bar{\xi}}} = \bar{\bar{\xi}}^3 - \hat{\xi} + N\bar{\bar{\xi}}^3\cdot\hat{\hat{\lambda}} - \frac{1}{2}\boldsymbol{c}\cdot\hat{\hat{\lambda}} \doteq \boldsymbol{0} \tag{19a}$$

and consequently to

$$\boxed{\begin{aligned} \bar{\bar{\xi}}^3 &= (I_2 + \hat{\hat{\lambda}}\cdot N)^{-1}(\hat{\xi} + \frac{1}{2}\boldsymbol{c}\cdot\hat{\hat{\lambda}}) = \\ &= \hat{\xi} - \frac{1}{2}(I_2 + \hat{\hat{\lambda}}\cdot N)^{-1}\boldsymbol{c}\cdot\hat{\hat{\lambda}}. \end{aligned}} \tag{19b}$$

The other necessary condition will then come from

$$\frac{\partial\Phi}{\partial\lambda} = \sigma_0^2 + (\bar{\bar{\xi}}^3)^T N(\bar{\bar{\xi}}^3 - \hat{\xi}) \doteq 0, \tag{20a}$$

which, after inserting (19b), leads to the polynomial

$$\begin{aligned} 2\sigma_0^2 - (\hat{\xi} + \frac{1}{2}\boldsymbol{c}\cdot\hat{\hat{\lambda}})^T(I_2 + \hat{\hat{\lambda}}\cdot N)^{-1}\cdot \\ \cdot N\cdot(I_2 + \hat{\hat{\lambda}}\cdot N)^{-1}\boldsymbol{c}\cdot\hat{\hat{\lambda}} = 0, \end{aligned} \tag{20b}$$

or

$$\boxed{\begin{aligned} 2\sigma_0^2 - (\hat{\xi} + \frac{1}{2}c\cdot\hat{\hat{\lambda}})^T(I_2 + \hat{\hat{\lambda}}N)^{-1}\boldsymbol{c} &= \\ = (\hat{\xi} + \frac{1}{2}\boldsymbol{c}\cdot\hat{\hat{\lambda}})^T(I_2 + \hat{\hat{\lambda}}N)^{-2}\boldsymbol{c}, \end{aligned}} \tag{20c}$$

which needs to be solved for $\hat{\hat{\lambda}}$. This Lagrange multiplier can then be re-implemented into (19b) to find $\bar{\bar{\xi}}^3$.

Point 4: Fourthly, we seek the proportionality factor α in $\bar{\bar{\xi}}^4 = \alpha\cdot\hat{\xi}$, which leads to the equation

$$\alpha^2\cdot(\hat{\xi}^T N\hat{\xi}) - \alpha\cdot(\hat{\xi}^T N\hat{\xi}) + \sigma_0^2 = 0 \tag{21}$$

and thus to the solution

$$\boxed{\alpha = \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4\sigma_0^2/(\boldsymbol{c}^T N^{-1}\boldsymbol{c})}.} \tag{22}$$

Note that α approaches 1 when σ_0^2 approaches 0.

3. An example

A suitable example will be presented at the JISDM 2019 conference.

4. Conclusions

In this paper, the reproBLE has been studied for single point adjustments in 2-D. Its location is defined by an *ellipse* as long as the existence condition (12) holds.

Among four different reproBLEs of special interest, only the third and fourth seem to be superior to the BLUUE in our example (but not in others). Hopefully, the comparison of their formal MSE matrices will provide more clarity.

Acknowledgements

Fruitful discussions with my colleagues Kyle Snow and Cuiping Guo have helped to shape the above material in the way it is here presented. This assistance is very much appreciated.

References

Gruber, M. (1998). *Improving Efficiency by Shrinkage: The James–Stein and Ridge Regression Estimators*. Dekker: New York/NY. 632 pp.

Schaffrin, B. (2000). “Minimum mean square error adjustment. Part 1: The empirical BLE and the repro-BLE for direct observations”. In: *J. of the Geodetic Soc. of Japan* 46(1), pp. 21–30.

Schaffrin, B. and P. Xu (2017). “Minimum mean squared error adjustment. Part 2: The Empirical BLE and the reproBLE for multivariate positioning”. Paper prepared for the IAG Symp. on Positioning and Applications, G05: Multi-signal positioning (Kobe/Japan).

Xu, P. (1998). “Truncated SVD methods for discrete linear ill-posed problems”. In: *Geophysical Journal International* 135(2), pp. 505–514.