# An Improved Hybrid Geoid Model over Kingdom of Saudi Arabia Utilizing New GNSS Ellipsoidal Heights on Benchmarks of KSA National Vertical Network

**Othman AL-KHERAYEF, Iurii GOLUBINKA, Sultan AL-SHAHRANI and Rossen GREBENITCHARSKY, Kingdom of Saudi Arabia**

**Key words**: Hybrid Geoid, KSA-GEOID2017, ellipsoidal heights, orthometric height

## SUMMARY

The most recent GNSS/leveling campaign in Kingdom of Saudi Arabia (KSA) took approximately two years (2017 – 2019) and provides as outcome around ~3500 ellipsoidal heights of benchmarks of KSA National Vertical Network (KSA-NVN). An improved version of current KSA-GEOID17of the Kingdom was determined based on common utilization of new GNSS/leveling data and terrestrial gravity measurements on KSA-NVN.
The applied hybrid geoid modelling procedure corresponds to the “classical” approach and was done according to the following consequence: (a) detecting and removal of systematic part from the data, based on rigorous Helmert transformation; (b) statistical analysis of geoid residuals after systematic part removal; (c) conversion of GNSS/leveling derived geoid heights into gravity anomalies; (d) forming difference between computed and observed terrestrial gravity anomalies; (e) statistical analysis of gravity differences (empirical and analytical covariance function construction); (f) transformation of derived gravity covariance function into geoid heights covariance function; (g) construction of transformation grid between gravimetric and hybrid geoids using least squares prediction of geoid differences. Validation of improved hybrid geoid has been conducted by utilizing additional gravity and geoid heights data and information over KSA and conclusions regarding applicability of the new improved version of KSA-GEOID17 have been drawn.

# An Improved Hybrid Geoid Model over Kingdom of Saudi Arabia Utilizing New GNSS Ellipsoidal Heights on Benchmarks of KSA National Vertical Network

**Othman AL-KHERAYEF, Iurii GOLUBINKA, Sultan AL-SHAHRANI and Rossen GREBENITCHARSKY, Kingdom of Saudi Arabia**

## 1. INTRODUCTION

For the present moment Kingdom of Saudi Arabia is taking actions for establishment and implementation a precise unified and homogeneous spatial reference system: Saudi Arabia Spatial Reference System (SANSRS). One of the major components of the SANSR is a precise local geoid model covering all territory of KSA with homogeneous accuracy not less than 2 cm.
The efforts that have been undertaken by the General Commission for Survey for this purpose resulted with the computation of geoid model KSA-GEOID2017.
In 2019 successful completion of the GPS/leveling project along the 3465 points of KSA National Vertical Network made possible to improve KSA-GEOID17and as a result to create new geoid model.
The main purpose of the paper is to present the methodology and results of the improvement of this hybrid geoid model for the Kingdom of Saudi Arabia.

## 2. METHODOLOGY FOR THE COMPUTATION OF IMPROVED GEOID MODEL FOR THE KSA (KSA-GEOID2017(I))

Given the availability of additional collocated height observations at some Vertical Reference Frame (VRF) and the existence of a geoid model over an area of interest, we can form proper observation equations combining the three height types: orthometric, ellipsoidal and geoid heights, in a least-squares adjustment aiming at the determination the new hybrid geoid surface well fitted to the this VRF.
If we will assume that we have available a network of BMs, at which we have available GNSS, orthometric with spirit levelling and geoid heights, for each point of the network (*i*) we can form a set of equations:

$$l_i = f_i + v_i^h + v_i^H + v_i^N \tag{1}$$

where $l_i$ – vector of observations; $v_i$ random errors not in the deterministic part, ellipsoidal, orthometric and geoid heights; $f_i$ – deterministic treatment of the differences between the two datums, and can be based on a parametric model:

$$f_i = a_i^T x_i \tag{2}$$

where $a_i$ - elements of design matrix and $x_i$ − uncnowns.
We can define the vector of observations as:

$$l_i = h_i - H_i - N_i \tag{3}$$

Based on the fact, that we can determine GNSS/leveling geoid height ($N_i^{GPS}$) as difference of GNSS geometric heights and levelling-based orthometric heights (Soycan, 2003), vector of observations will be:

$$l_i = N_i^{GPS} - N = h_i - H_i - N_i \tag{4}$$

Based on this observation equation we can follow the least square adjustment (LSA) algorithm and determine the deterministic transformation between the two models. The main disadvantage of this approach, is that the differences between the two geoids cannot be attributed to some simple bias and tilt. Inherent in their differences, especially due to the different frequencies modelled by the two, are residual stochastic signals that cannot and should not be modelled as deterministic ones with a simple model of the form $a_i^T$. In order to achieve a more rigorous modeling of their differences, we should employ a hybrid deterministic and stochastic approach, during which we simultaneously estimate stochastic signals ***s*** during the adjustment process, therefore we expand the least-squares problem into an adjustment with stochastic parameters (Kotsakis & Sideris, 1999). In order to simultaneously estimate the deterministic parameters and the stochastic ones we will use a mixed adjustment scheme with observation equations. According to (Kotsakis & Sideris, 1999), (Grebenitcharsky et al., 2005) the stochastic part of the signal is contained in the original observation equation of the signal, which now takes the form:

$$b_i = (a_i^T x + s_i) + v_i = a_i^T x + e_i \tag{5}$$

Which in matrix form is:

$$b = Ax + s + Kv \tag{6}$$

where $K$ is an identify matrix which in our case becomes $(h - H - N) = 0$ $K = [I_n, -I_n, -I_n]$, $n$ being the number of observations. The minimization criteria for the LSA problem becomes:

$$s^T Q_s^{-1} s + v_h^T Q_h^{-1} v_h + v_H^T Q_H^{-1} v_H + v_N^T Q_N^{-1} v_N = min \tag{7}$$

where $Q_s^{-1}$ is properly selected weight matrix for the unknown signal. The solution is similar to the regular observation equation adjustment problem:

$$P = (Q_h + Q_H + Q_N + Q_s)^{-1} \tag{8}$$

$$\hat{x} = (A^T P A)^{-1} A^T P b \tag{9}$$

$$\hat{v} = b - A\hat{x} \tag{10}$$

We assume that there are no errors in the estimation of the unknown stochastic signals $\hat{s}$, or if there are, then their magnitude is minimal (Grebenitcharsky et al., 2005). Moreover, we assume that the errors of the observations are contained totally in the estimation of their errors. In the above equations (8) we do not known the weight matrix $Q_s$, nor do we have an estimate of the signal ***s*** in order to estimate the empirical covariance function. Therefore, the adjustment is performed sequentially, during which process we make a "smooth" selection for the covariance matrix (and consequently the weight matrix) of the unknown signal to be equal to the unity matrix $(Q_s = I_n)$. This selection can be regarded as the smoothest one fitting optimally in the available observations ***b***, the selected parametric model $a_i^T x_i$ and the stochastic model of the observations $(Q_h, Q_H, Q_N)$ (Grebenitcharsky et al., 2005). Using this initial approach, the reach a first solution for the unknown signal as:

$$W = I_n - A(A^T(Q_h + Q_H + Q_N + I_n)^{-1}A)^{-1}A^T(Q_h + Q_H + Q_N + I_n)^{-1} \tag{11}$$

and

$$\hat{s}_{init} = (Q_h + Q_H + Q_N + I_n)^{-1}Wb \tag{12}$$

The next step is to estimate the general trend in the signals ***s*** with the adjustment of smooth corrector surface to the signals $\hat{s}_{init}$. By estimating the differences $\boldsymbol{\hat{m}s}$ that this corrector suface gives at the network BMs, we can construct reduced values for both the observations and the signal, i.e. :

$$b_r = b - \widehat{m}_s \tag{13}$$

and

$$s_r = s - \widehat{m}_s \tag{14}$$

In this case, i.e., after removing a smooth corrector surface, we can safely assume that the reduced signals have a mean close or equal to zero, hence they are unbiased and stochastic random processes, i.e., that $(E\{s_r\} = 0)$ (Grebenitcharsky et al., 2005). In this case, we cannot proceed with the usual collocation approach and estimate an empirical covariance function for the reduced signals that will describe their statistical characteristics and will be used for the estimation of the signal covariance matrix $C_s$ and consequently of the weight matrix $Q_s$. Note that as in collocation, we will use the empirical covariance function to fit a selection of analytical covariance function models to it, so as to select the most proper one. Using the new improved stochastic model, we can now derive new optimal estimates of the unknown signals as:

$$W = I_n - A(A^T(Q_h + Q_H + Q_N + I_n)^{-1}A)^{-1}A^T(Q_h + Q_H + Q_N + Q_{sr})^{-1} \tag{15}$$

$$\hat{x} = (A^T(Q_h + Q_H + Q_N + I_n)^{-1}A)^{-1}A^T(Q_h + Q_H + Q_N + Q_{sr})^{-1}b_r \tag{16}$$

$$\hat{s}_r = Q_s(Q_h + Q_H + Q_N + Q_s)^{-1}Wb_r \tag{17}$$

and individual errors of the observations can be determinaed as:

$$\hat{v}_h = Q_h(Q_h + Q_H + Q_N + Q_s)^{-1}Wb_r \quad (18)$$

$$\hat{v}_H = -Q_H(Q_h + Q_H + Q_N + Q_s)^{-1}Wb_r \quad (19)$$

$$\hat{v}_N = -Q_N(Q_h + Q_H + Q_N + Q_s)^{-1}Wb_r \quad (20)$$

Therefore, the estimation the unknown deterministic and stochastic signals can be summarized as:

$$\hat{x} = (A^T PA)^{-1}A^T Pb \quad (21)$$

and

$$\hat{s} = C_{sl}\bar{C}^{-1}(l - AX) \quad (22)$$

where $C_{sl}$ is the cross-covariance matric between the signal and the observations, $\bar{C} = C + D$ consists of the covariance matrix of the signal *C* and of their errors *D*. The latter is usually assumed equal to the identity matrix, as no information about the cross-covariance error matrix of the ellipsoidal, orthometric and geoid heights is available. Therefore, we will assume that $D = \sigma_0^2 I$, where $\sigma_0$ is the a-priori standard deviation of the observations.
The crucial point in this stochastic estimation is the proper selection of an analytical model of the signal covariance function, based on the empirical one, which will describe rigorously the statistical characteristics of the signals and will provide reliable estimates. Within this study, we will investigate the following analytical covariance function models for the geoid height differences:
2nd-order Gauss–Markov:

$$K_N(\rho) = K_0(1 + A\rho)e^{-A\rho} \quad (23)$$

3nd-order Gauss–Markov:

$$K_N(\rho) = K_0\left(1 + A\rho + \frac{A^2\rho^2}{3}\right)e^{-A\rho} \quad (24)$$

Logarifmic (Grebenitcharsky et al., 2005):

$$K_N(\rho) = K_0 \ln\left(\frac{2(e-1)}{1 + \sqrt{1 + A^2\rho^2}} + 1\right) \quad (25)$$

where $K_0$ is the variance of the observations $l_i$; *A* is a parameter related to the correlation length; $\rho$ is the distance in km. The corresponding covariance functions for the gravity anomaly differences are as follows:

2nd-order Gauss–Markov:

$$C(\rho) = C_0 \left(1 - \frac{A\rho}{2}\right) e^{-A\rho} \tag{26}$$

3rd-order Gauss–Markov:

$$C(\rho) = C_0 \left(1 + A\rho - \frac{A^2\rho^2}{2}\right) e^{-A\rho} \tag{27}$$

Moritz 1980:

$$C(\rho) = \frac{C_0}{(1 - A^2\rho^2)^{3/2}} \tag{28}$$

The consistency of the analytical models of (23) & (26), (24) & (27), has been investigated in (Jordan,1972), while the covariance model of (28) is one possible choice of a positively definite function for the approximation of an empirically derived covariance function of the gravity anomaly difference. The derivation of (25) for the geoid height differences is discussed in (Grebenitcharsky et al., 2005), while the same authors discuss the methodological scheme followed to compute the analytical covariance functions (ibid.).

## 3. DATA USED IN THIS STUDY

This study used two data sets available for the KSA territory:

- Geoid model KSA-GEOID2017.
- Network of 3652 GNSS/leveling points.

### 3.1 KSA-GEOID2017

KSA-GEOID17model (see Figure 1) is based on the EIGEN6C4 reference field (incorporating GOCE and GRACE satellite data), new DTU15 satellite altimetry data offshore, and more than a half million gravity data points. The geoid is fitted to the new KSA Vertical Reference Frame (KSA-VRF14) through a set of 280 GNSS/levelling points along the new GCS 1st order-levelling network (Al-Kherayef O., Grebenitcharsky R, 2019).
The fitted geoid is estimated on a grid of 0.01° x 0.015° resolution to have errors of 2 cm r.m.s. in the eastern region (where underlying gravity data region is good), and 10-20 cm r.m.s. in the western regions, where only sparse gravity data are available. The largest uncertainties are in mountains and coastal plains along the Red Sea. When there is a change in the geoid slope and the sparse gravity data area, the GPS station with orthometric height is required (Al-Kherayef, 2015).

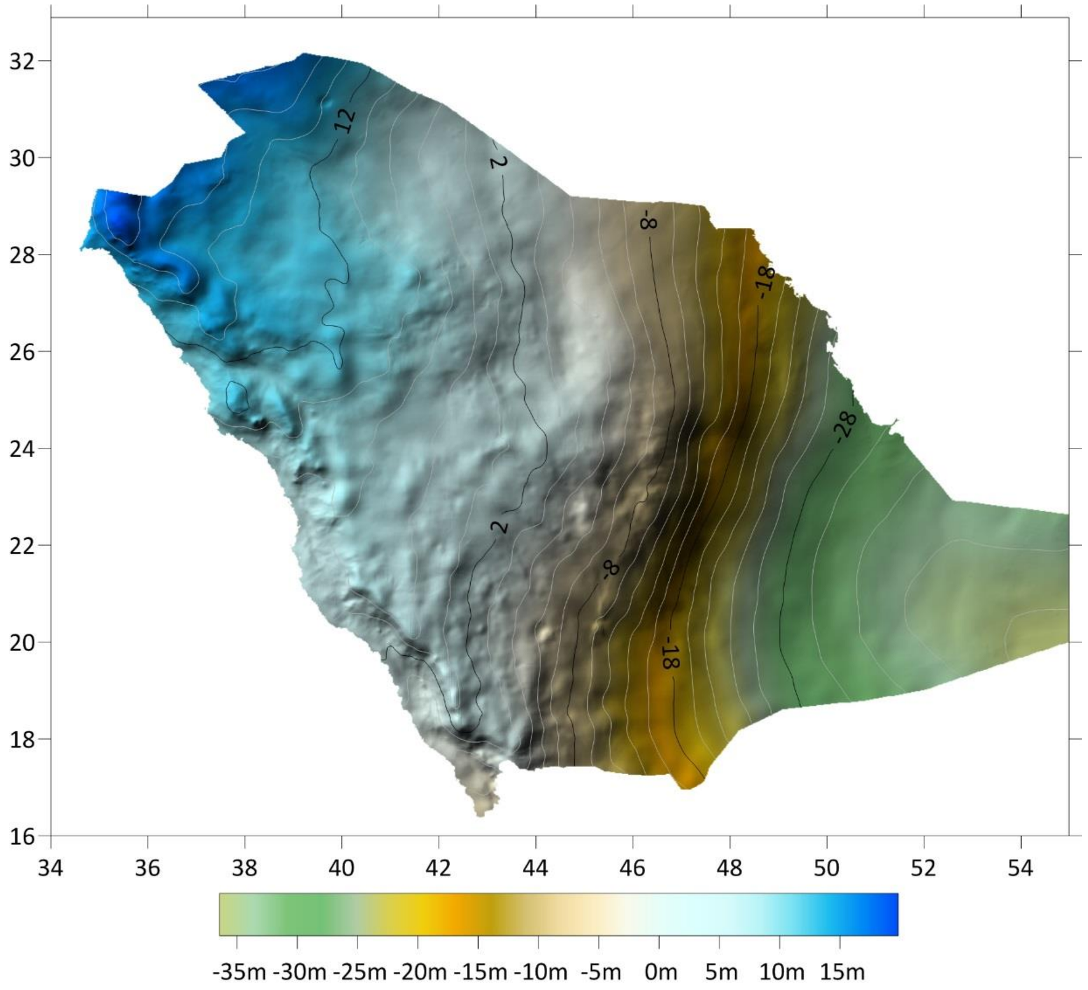



**Figure 1:** KSA-GEOID17model (Al-Kherayef O., Grebenitcharsky R, 2019)

### 3.2 GNSS/LEVELING NETWORK

GNSS/leveling project was conducted during 2017 – 2019 on the 3652 points (see Figure 2) of the National Vertical Network (3397 of leveling benchmarks and 255 points of the primary geodetic network). After the data quality check the total number of utilized points was 3465.


**Figure 2:** GNSS/leveling points

As a result of the project, the ellipsoidal heights of the points have been computed with absolute accuracy of 1.5 – 2.0 cm. Each point is provided by the precise value of orthometric heights, related to the Jeddah14, which is the latest KSA Vertical Reference Frame.
The distribution of the new GNSS/leveling data is much more homogeneous in comparing to the previous set of points used for KSA-GEOID17determination, and the accuracy of the hybrid geoid is largely depends on the number of GNSS/leveling points and their spatial distribution (Erol, 2004). According to this, we expect significant improvement of the accuracy of the new geoid model comparing to the KSA-GEOID2017.

## 4. DETERMINATION OF THE IMPROVED GEOID MODEL: KSA-GEOID2017(I)

The methodology, described in the previous chapter has been used to improve existing geoid model KSA-GEOID17by its fitting to new Jeddah2014 VRF system through a set of 3465 GNSS/leveling points. The resulted model KSA-GEOID17(i) has been computed on grid of 0.01° x 0.015° resolution (see Figure 3).

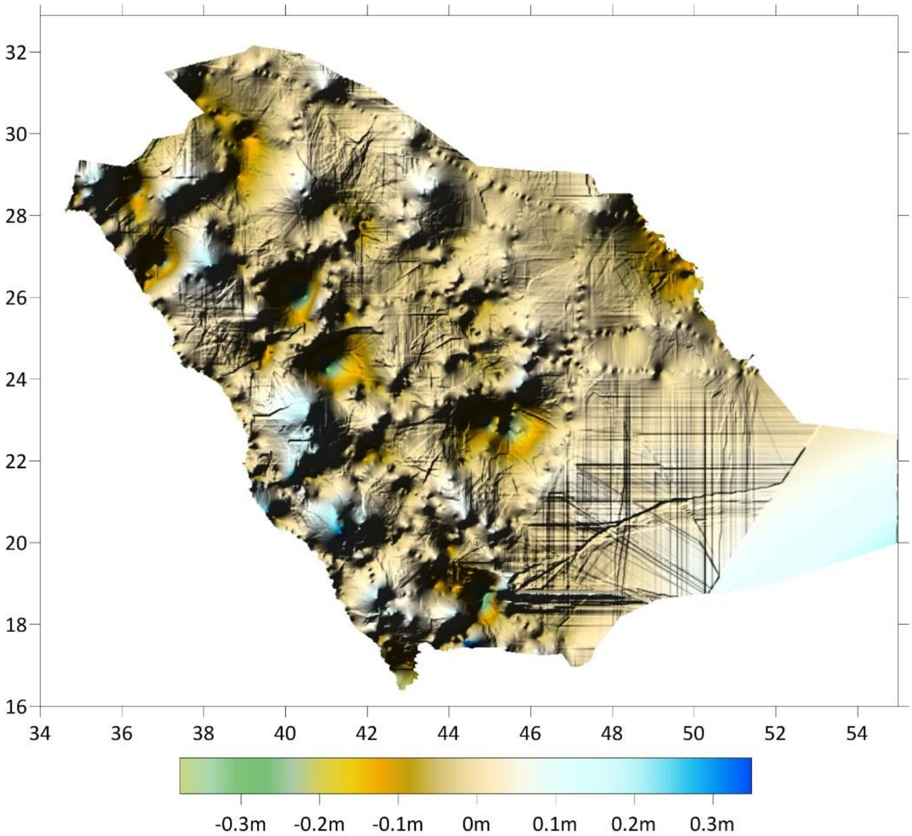



**Figure 3:** Differences between KSA-GEOID17& KSA-GEOID2017(i)

For the purpose of evaluation of accuracy of the resulted model, we have compared modeled geoid heights for a set of GNSS/leveling points:

$$dN_i^{gm} = N_i^{gm} - N_i^{GNSS}, \tag{29}$$

where subscript *gm* corresponds, to the geoid model, *i* – to the GNSS/leveling point, $N_i^{GNSS}$ is computed as difference of ellipsoidal and orthometric heights:

$$N_i^{GNS} = h_i - H_i \tag{30}$$

In addition, same comparison has been done for the previous model KSA-GEOID17. Figure 4 shows distribution of the differences for both models, for each point, and Table 1 gives an overview for a statistical information for differences for the both models.

**Table 1** Statistics for differences between geoid heights from GNSS/leveling and from KSA geoid models

| Parameter | KSA-GEOID17 | KSA-GEOID17(i) |
|---|---|---|
| Min (m) | -0.3881 | -0.2492 |
| Max (m) | 0.3586 | 0.1540 |
| Mean (m) | -0.0117 | -0.0227 |
| STD (m) | 0.0738 | 0.0363 |
| RMS (m) | 0.0747 | 0.0428 |

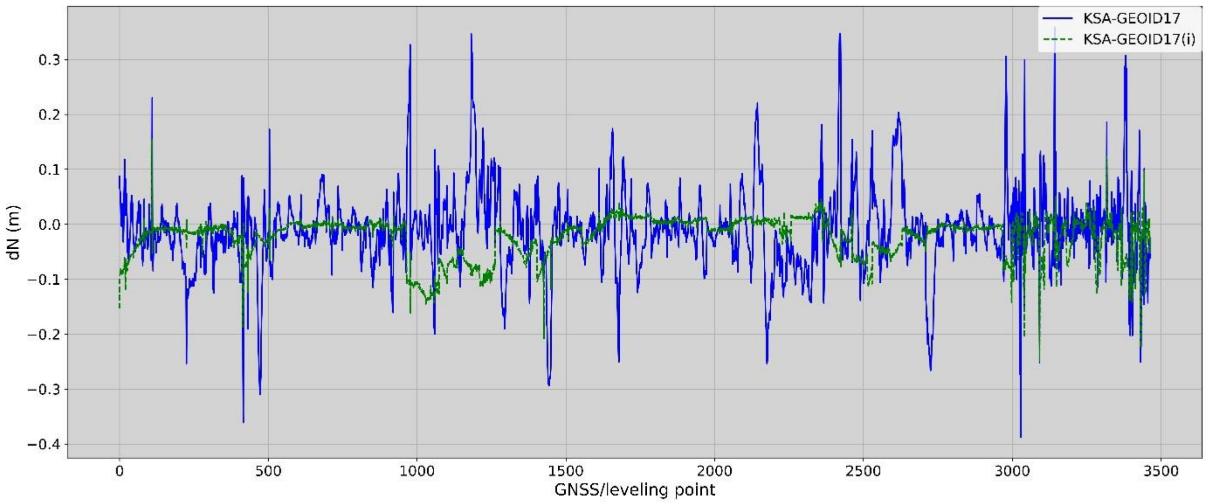
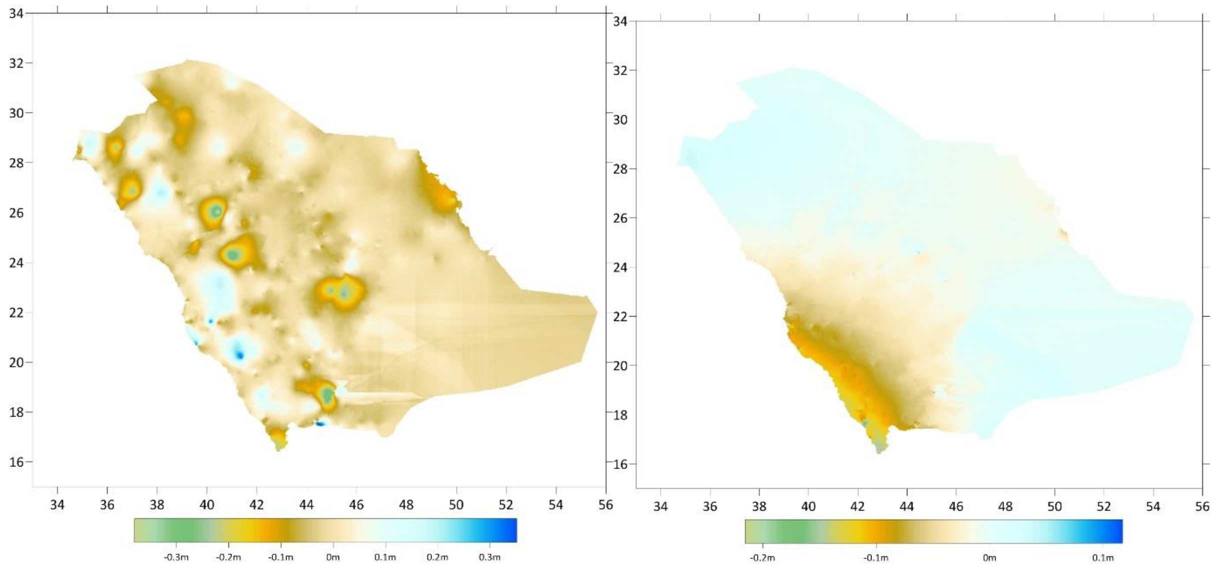



**Figure 4:** Distribution of the geoid heights differences between GNSS/leveling and KSA geoid models

From the Figure 4 and Table 1 we can see significant improvement of the KSA-GEOID17(i) in comparing to the previous model. Spatial distribution of the differences of given in Figure 5.



a) b)

**Figure 5:** Spatial distribution of the geoid heights differences (in meters) between GNSS/leveling and *a)* KSA-GEOID17; *b)* KSA-GEOID17(i)

Differences of the geoid heights ($N$) computed using two models (KSA-GEOID17 and KSA-GEOID17(i)) by each point of GNSS/leveling is shown in Figure 6.

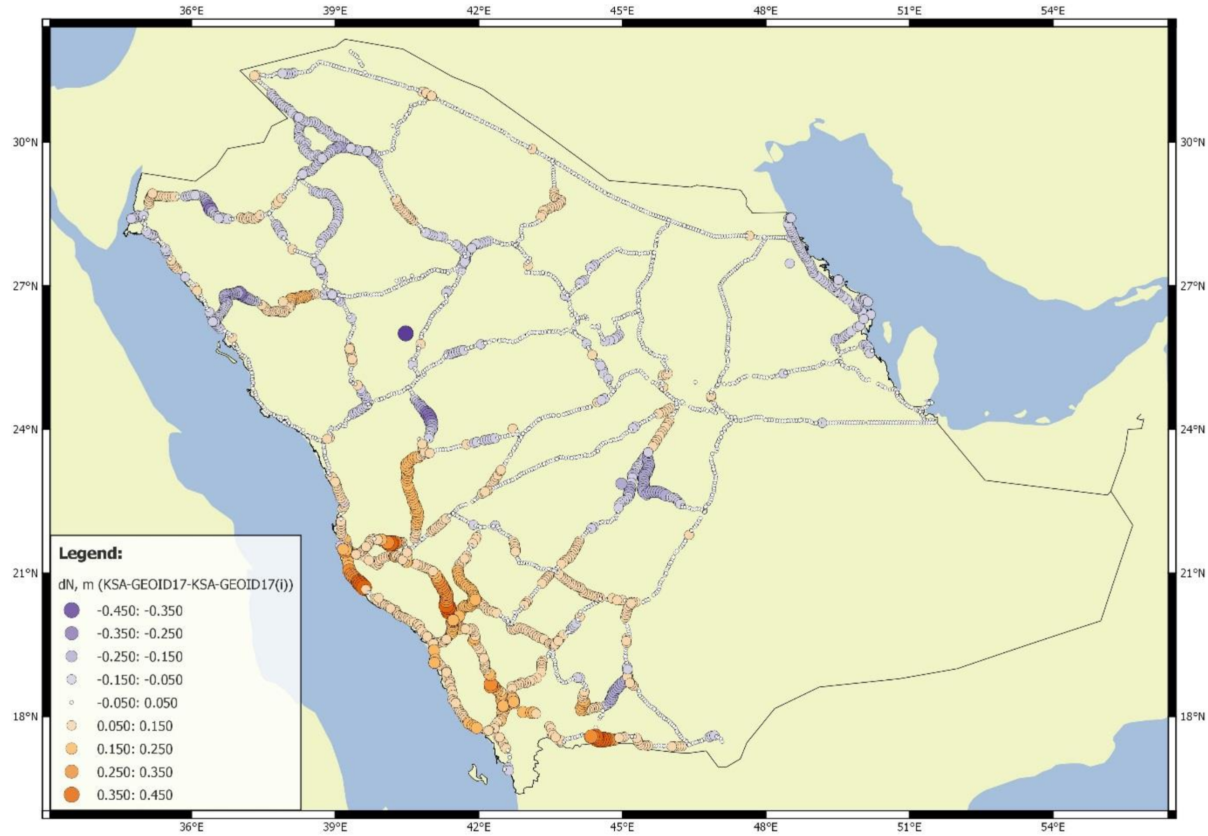



**Figure 6:** Differences between geoid heights (*N*) computed using two models (KSA-GEOID17 and KSA-GEOID17(i)) by each point of GNSS/leveling

From the Figure 6 we can clearly identify the large area of uplift (positive differences of geoid heights) in the southwestern part of the Kingdom and as well some local areas of subsidence (negative differences of geoid heights) in central and northern part of KSA. For the rest of the country differences are within ± 5 cm. The southwestern uplift may be related to some regional gravity anomalies and needs further investigations.

In addition, differences in geoid heights for both models were grouped by 88 leveling lines. Plot showing mean difference and its standard deviations (±σ) by the each line is given in Figure 7, and Figure 8 shows the spatial distribution of the differences.

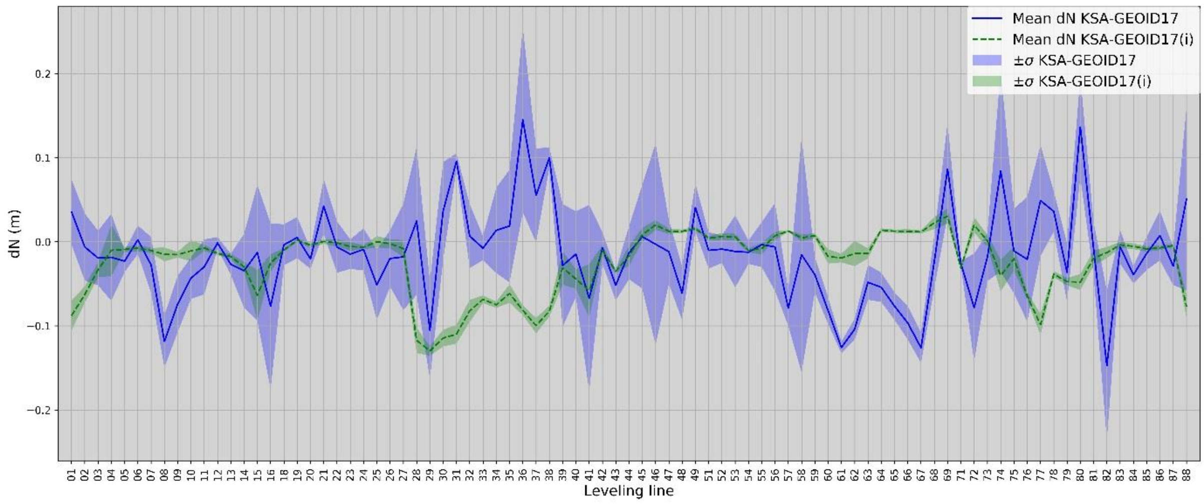



**Figure 7:** Distribution of the geoid heights differences between GNSS/leveling and KSA geoid models by the leveling lines


**Figure 8:** Spatial distribution of the geoid heights differences between GNSS/leveling and KSA geoid models by the leveling lines

Figure 8 confirms the improvement of the new geoid model, especially in the western part of the country where all lines are provided with much smaller values of standard deviations of the geoid heights differences. In average Standard Deviation (STD) of *dN* for entire Kingdom is less than 1 cm, but despite this values of STD's for three lines in western, southwestern, and central part of the country are larger than 2 cm (leveling lines: 04, 15, 41).

## 5. CONCLUSIONS

- The utilization of the described methodology allowed us to compute new improved geoid model for the Kingdom of Saudi Arabia: KSA-GEOID17(i), which is more accurate comparing to the previous one (KSA-GEOID17).
- Geoid heights of both models were compared between each other, and with the geoid heights computed for the 3465 GNSS/leveling points. Result of the comparison showed significant improvement of new model: STD of the geoid heights differences is two times smaller for KSA-GEOID17(i).
- Evaluation of the computed model by each leveling line showed that it is more accurate (in average STD of KSA-GEOID17(i) is 5 times smaller than previous one) and this accuracy is homogeneously distributed over the Kingdom, with the exception of the southeastern part, where no data are available.
- In addition comparison of two models showed significant uplift in the southwestern part of the country, which may be related to gravity anomalies in this area and needs further investigation.
- In future this studies will be continued and the presented geoid model will be improved. Now GCS is conducting the project for airborne gravity, which will allow to fill gaps for gravity data within entire the Kingdom, especially in the Southern part of the country.

## REFERENCES


- Al-Kherayef O. (2015) Re-Establishment of Geodetic Networks at General Commission for Survey. FIG Working Week 2015. Sofia, Bulgaria, 17-21 May 2015.
- Al-Kherayef O., Grebenitcharsky RS (2019) The Kingdom of Saudi Arabia Geodetic and Vertical Reference Frames –the main Components of National Spatial Reference System. EUREF 2019 Symposium, 23-25 May 2019.
- Erol B, Celik R (2004), Precise local geoid determination to make GPS technique more effective in practical applications of geodesy.. In FIG working week, (s. (Vol. 17, No. 3, pp. 22-7).)
- Grebenitcharsky RS, Rangelova EV, Sideris MG (2005) Transformation between gravimetric and GPS/levelling-derived geoids using additional gravity information. J Geodynamics 39(527-544). doi: 10.1016/j.jog.2005.04.008.
- Jordan SK (1972) Self-consistent models for the gravity anomaly, vertical deflections, and undulations of the geoid. J Geoph Res 77 (20), 3660–3670.
- Kotsakis K and Sideris MG (1999) On the adjustment of combined GPS/levelling/geoid networks. J Geodesy 73:412–421.
- Moritz H (1980) Advanced Physical Geodesy. H. Wichmann, Karlsruhe, Germany.

- Soycan, M. E. (2003). Surface modeling for GPS-levelling geoid determination Newton's Bulletin., (s. 1, 41-52.)


## BIOGRAPHICAL NOTES

## CONTACTS


Title Given name and family name: **Sultan AL-SHAHRANI**
Institution: General Commission for Survey (GCS)
Address: P.O.Box: 87918
City: Riyadh
COUNTRY: Kingdom of Saudi Arabia
Tel. +966533111554
Email: sf.alshahrani@gcs.gov.sa
Web site: https://www.gcs.gov.sa/