Investigation of the Use of the Ellipsoidal Normal to Model the Plumb Line in a Millimeter Cadastre

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ABSTRACT

It may soon become possible to routinely stake an absolute location of a geodetic coordinate repeatedly on the ground to millimeter accuracy using GPS. A cadastre could be developed using such coordinates as the definitive location of boundary corners. A boundary line would be a one-dimensional line connecting boundary corners. A boundary line extending into space or toward the center of the Earth would be a two-dimensional surface with length, height and no width. A cadastre using geodetic coordinates would most naturally use the ellipsoidal normal to define the vertical dimension, however historically (and in many jurisdictions, legally) the vertical dimension has been defined by the plumb line. Unlike the ellipsoidal normal, the plumb line is not a straight line but is rather a space curve with finite radius of curvature and torsion (Leick 1995). The deflection of the vertical is the angular difference at the Earth's surface between the ellipsoidal normal and the tangent to the plumb line. (For instance, a 2 second deflection of the vertical results in about 5 cm difference in a vertical distance of 3 km underground and 2 km above ground, distances over which an owner might reasonably be expected to have actual control). This paper will investigate the use of the ellipsoidal normal to model the plumb line with respect to the vertically extended boundary line in which it is proposed to be a 3-D surface that has length, height and enough width to include all variations caused by the space-curved plumb line.

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1. INTRODUCTION

With the continuing advances in technology of the Global Positioning System (GPS), the day will soon come when the absolute location of a geodetic coordinate (latitude, longitude) may be repeatedly staked on the ground to one-millimeter accuracy. Many jurisdictions allow a geodetic coordinate to definitively and legally mark a boundary corner (For instance, Massachusetts General Law Chapter 97 §17 states:

> *For the purpose of describing the location of any …land boundary point in the commonwealth it shall be a complete, legal and satisfactory description of such location to give the position of said…land boundary point on the Massachusetts Coordinate System.*

See also Maine Revised Statutes Title 33 Chapter 13 §803-A). Therefore, in the future a cadastre may be developed in which boundary corners of ownership are definitively located on the Earth's surface not by physical monuments but by geodetic coordinates. However, ownership of land does not just include the Earth's surface and what is attached to it, but also the space beneath and above the surface (Creteau 1977). A computer screen showing a two-dimensional surface where geodetic coordinates mark boundary corners might model a cadastre of such three-dimensional parcels of land. In such a model, the third dimension of height might be modeled by the ellipsoidal normal; however, traditionally and legally the plumb line has been used to define this third dimension. This paper will analyze the use of the ellipsoidal normal to model the plumb line.

2. CONCEPT OF OWNERSHIP

The term "land" includes not only the soil and everything attached to it, but also the space beneath and above the surface. The classic concept is that ownership of land includes not only the surface of the Earth but extends infinitely upward and downward. This concept has been modified in recent times so that ownership of land includes the space above and below the Earth's surface, but only to the extent necessary for the enjoyment and exploitation of the property (Creteau 1977). Thus planes flying over a parcel of land are not encroaching on the rights of the landowner because the planes are flying above where the landowner might reasonably be expected to have actual control. For instance, in the Code of Federal Regulations, 14 CFR- Chapter I Part 91 Section 91.119 states:

> *Except when necessary for takeoff or landing, no person may operate an aircraft below the following altitudes:*

…

(b) <u>Over congested areas:</u> Over any congested area of a city, town, or settlement …an altitude of 1000 feet above the highest obstacle within a horizontal radius of 2000 feet of the aircraft.

(c) <u>Over other than congested areas:</u> An altitude of 500 feet above the surface, except over open water or sparsely populated areas. In those cases, the aircraft may not be operated closer than 500 feet to any person, vessel, vehicle, or structure.

…

The creator of a cadastre must therefore determine the limits of ownership above and below the Earth's surface. For discussion purposes I will assume that a reasonable limit will be 500 meters above and below the surface of the Earth.

3. DEFINITION OF WORDS DESCRIBING LAND OWNERSHIP

American Jurisprudence (12 Am Jur 2d Boundaries § 53) (Lawyers Cooperative Publishing, 1962) states:

In the absence of any specific statutory provision governing the manner of measurement of distances, distance is to measured along the shortest straight line, on a horizontal plane…

Webster's dictionary (Webster 1986) has the following definitions:

Horizontal: *of or relating to the horizon*
Horizon: *1b2) The circle in which a plane perpendicular to the direction of gravity intersects the celestial sphere. 1b3) The plane tangent to the Earth's surface at the observer's location.*
Vertical: *2a) Perpendicular to the plane of the horizon.*

Combining the two definitions of "horizon" gives the definition of "horizontal" as "of or relating to the plane perpendicular to the direction of gravity at the Earth's surface".

The **plumb line** is the space curve that is always tangent to the direction of gravity and connects a point on the geoid to a point on the surface of the Earth (Smith 2001). The direction of gravity depends on the gravity potential field of the Earth, which changes for the following periodic and secular reasons (Smith 2001):

Lunar and solar tides on the oceans, atmosphere and lithosphere
Seasonal shifts in the water table
The falling of leaves, and growing of new leaves in fall and spring (small, probably barely measurable, but an unquestionable seasonal shift in the mass distribution of the Earth)
Plate tectonics
Glacial rebound

Thus for a specific point on the Earth's surface, the direction of the plumb line may change with elevation and may also change with time. Therefore the horizontal plane of a point at the Earth's surface may have a different direction than the horizontal plane for that point at an elevation other than at the surface, and further the direction of these horizontal planes may change with time.

Defining the vertical dimension of a boundary line by using this time and elevation dependent plumb line results in a dynamic complex shape for a boundary line that is extended above or below the Earth's surface. A "straight" boundary line on the Earth's surface would be extended above and below by the plumb line that exists at each point of the boundary line. Rather than the two-dimensional boundary line being a plane having length and height, it will be a three-dimensional wavy plane looking much like a blanket hung vertically with a gentle breeze making it ripple.

Based on my research, I find no statutory or case law that gives a legal definition of "horizontal" (although there may be common law that has developed over the years). A cadastre based on the above-described literal definition of "horizontal" may have boundary lines that are fixed at the surface of the ground but are continually changing in elevation and time. The law-making organizations and courts may want boundary lines to be fixed in time. For instance, the legal establishment may find intolerable the fact that tall structures may encroach over the boundary line at one time but not at another because of the dynamic aspect of boundary lines. They may also find intolerable the fact that tall structures encroach at some elevations but not at others. The legal establishment may want to simplify the definition of "horizontal" so that boundary lines are fixed and easy to determine. If so, then the following definition is possible.

The definition of "horizontal" offered in this paper is defined by the direction of gravity at the Earth's surface. One simplification would be to determine the "average" direction of gravity at the boundary corner at the Earth's surface over time, and make that direction a straight line as we go above or below the Earth's surface, rather than use the space curve of the plumb line. This paper will use this new definition of "horizontal". (Note that the legal definition can only be determined by future statutes and case law.)

4. MODELING THE CADASTRE

The cadastre is therefore defined as parcels of land ownership whose boundary corners are defined by geodetic coordinates on the Earth's surface. This boundary corner extends beneath and above the surface by a straight line that is tangent to the "average" direction of gravity that exists at the Earth's surface at that point. Each point along a boundary line would have its own vertical straight line that is tangent to the "average" direction of gravity at that point. The result will be a cadastre where a boundary line is a three-dimensional object made up of all vertical straight lines that are tangent to the direction of gravity that exists at each individual point along the boundary line. The boundary line will look like a rippling blanket that is frozen and unchanging. (For purposes of this paper, the cadastre will show land ownership for 500 meters above and below the Earth's surface.)

Our task is therefore to model the above-described cadastre. The requirement will be that the model must provide a two-dimensional view of the three-dimensional boundary line so that the model can be viewed on a computer monitor or shown on a two-dimensional paper map. Possible coordinate systems for use in the model will be shown next and then a proposed model will be introduced.

4.1 Ellipsoidal Surface Model

The ellipsoidal surface model coordinate system has units of latitude (φ), longitude (λ), and height above the ellipsoid (h). The reference ellipsoid most used is the Geodetic Reference System of 1980.

4.1.1 Advantages of Ellipsoidal Surface Model

The advantage of this system is that ellipsoidal height has a direction (ellipsoidal normal) that runs generally toward the center of the Earth and thus two-dimensional coordinates of latitude and longitude can generally model a three-dimensional land parcel.

4.1.2 Disadvantages of Ellipsoidal Surface Model

One disadvantage is that the direction of the ellipsoidal normal and the tangent to the direction of gravity are generally not the same. Another disadvantage is that the ellipsoidal surface is not a plane (note that by legal definition, distances are to measured on a plane).

4.2 Three-Dimensional Geodetic Model

This model is an extension of the ellipsoidal surface model in which a plane called the local geodetic horizon is created at a location with known geodetic coordinates (φ_o, λ_o, h_o) (Leick 1995). The local geodetic horizon has coordinates of (n, e, h). The n axis points north, the e axis points east, and the h axis coincides with the ellipsoidal normal at the known geodetic coordinate. (Note that the ellipsoidal height of the ellipsoidal surface model and the third axis of the local geodetic horizon both use the same symbol h even though they are different quantities. Thus in the above description, the local geodetic horizon has $h = 0$ when the ellipsoidal height of the defining point is h_o.)

4.2.1 Advantages of the Three-Dimensional Geodetic Model

One advantage is that the local geodetic horizon is a plane (note that legally, distances are to be measured on a plane). Another advantage is that the local geodetic plane can be transformed to be horizontal at the defining geodetic coordinate by rotating the h axis (which is coincident with the ellipsoidal normal at the defining geodetic coordinate) so that it becomes coincident with the tangent to the direction of gravity at the defining geodetic coordinate. (For land parcels with areas less than one square mile, the relative change in the direction of gravity for various boundary corners is usually negligible. Thus for all practical purposes, the transformed local geodetic plane is perpendicular to the direction of gravity for all of the said boundary corners.) A third advantage is that distances and areas figured from the local geodetic coordinates closely match those that exist at the surface of the Earth since the local geodetic horizon and the Earth's surface are coincident at the defining geodetic coordinate.

4.2.2 Disadvantages of the Three-Dimensional Geodetic Model

One disadvantage is that planes in general can only accurately model the Earth's surface for a finite region. Thus the further a boundary corner or line is from the defining geodetic coordinate, the more inaccurate local geodetic coordinates will be in modeling the actual geodetic coordinates. A second disadvantage is that typically each land parcel could have its own local geodetic horizon defined by the defining geodetic coordinate (φ_o, λ_o, h_o) of one of its boundary corners. Thus it would be difficult to model a large cadastre in two dimensions using this system. A third disadvantage is that vertical directions at all points on the local geodetic horizon are parallel, where all boundary corners of a land parcel converge at the Earth's center and diverge as they go out into space.

4.3 Conformal Mapping Plane Model

Conformal mapping is an extension of the ellipsoidal surface model system in that the ellipsoidal surface is conformally mapped onto a conformal mapping plane (In conformal mapping the angle between the tangents of curves on the ellipsoid and the angle between tangents of the mapped curves are preserved.). Examples of conformal mapping planes in the United States are the various zones of the State Plane Coordinate System of 1983 in which each uses the Transverse Mercator Model, the Lambert Conformal Model, or the Oblique Mercator Model.

4.3.1 Advantages of the Conformal Mapping Plane Model

One advantage is that the model is a plane. Another advantage is that State Plane Coordinate models are relatively well known and used throughout the United States.

4.3.2 Disadvantages of the Conformal Mapping Plane Model

The main disadvantage is that the conformal mapping plane model is only two dimensional, and does not have a vertical component that is inherent to the model. The conformal mapping model only maps points between the mapping plane and the surface of the ellipsoid. (The vertical component of ellipsoidal heights or orthometric heights is often used, but both of these systems of vertical heights are not directly tied to the conformal plane.) Thus a three-dimensional wedge of land cannot be directly modeled with a conformal mapping plane model.

5. PROPOSED MODEL: ELLIPSOIDAL SURFACE MODEL WITH FUZZY BOUNDARIES

This proposed model is based on the ellipsoidal surface model and uses the ellipsoidal normal to model the direction of the "average" direction of gravity at a point on the surface of the ground. The relationship between the ellipsoidal normal and the "average" direction of gravity is defined as the Deflection of the Vertical (DOV). (The National Geodetic Survey (NGS) has developed a database and software program called DEFLEC99 that can calculate the DOV for a point with known (φ, λ, h). The DOV is made up of a component *xi* (ξ) along the meridian, and a component *eta* (η) along the prime vertical.) The use of the ellipsoidal normal instead of the direction of gravity will introduce error in the model. This error will be manifested in this model through the use of a "fuzzy" boundary line that will have a width sufficient to contain the said error. Figure 1 describes this proposed fuzzy boundary system with an example.

Figure 1 describes the proposed fuzzy boundary system for a possible boundary point (named Crestone) located in Colorado. This point was scaled off of the Crestone Quadrangle 7.5 minute series topographic map, which was prepared in 1982 by the United States Geological Survey. The scaled coordinates of point Crestone on the Earth's surface on the said map have a latitude of North 38 degrees, 00 minutes, 00.00000 seconds, a longitude of West 105 degrees, 37 minutes, 30.00000 seconds and an ellipsoidal height of 3765 meters (The USGS map uses orthometric heights, which can be converted to ellipsoidal heights by applying the geoid undulation *N*. For this point an orthometric height of 3780 meters was scaled off the map. The NGS software program calculated an approximate N = -15 meters and thus the ellipsoidal height is about 3765 meters.) Based on these geodetic coordinates, the DEFLEC99 program gives DOV with components ξ = -6.77 seconds and η = -16.69 seconds. Using these components, the maximum DOV for this point is about 18 seconds. Assuming the limits of ownership to be 500 meters above and below the Earth's surface, the fuzzy boundary width would be 0.088 meters ($2 * 500 * \tan(18'') = 0.088 meters$). Note that the plumb line intersects the ellipsoid about 0.329 meters from the ellipsoidal normal intersection.

Figure 2 shows the xi (ξ) component in the plane of the meridian. Note that the plumb line and ellipsoidal normal that intersect at point Crestone at the Earth's surface are 0.124 meters apart at the ellipsoid surface. The ellipsoidal normal that marks the intersection of the plumb line with the ellipsoid has latitude of North 38 degrees, 00 minutes, 00.00398 seconds. ($\Delta\phi = \dfrac{\Delta northing}{M}$ where M is the radius of curvature of the ellipsoidal meridian for that point:

$M = \dfrac{a(1-e^2)}{(1-e^2\sin^2\phi)^{\frac{3}{2}}}$ For GRS 80, $a = 6378137 meters$ exactly and $e^2 = 2f - f^2$ where $\dfrac{1}{f} = 298.257222101$ and $\phi = 38^o$. Thus $M = 6402490.355 meters$ and $\Delta\phi = \dfrac{0.124}{M}$ $= 0.00398 \sec$)(Leick 1995)

Figure 3 shows the eta (η) component in the plane of the prime vertical. Note that the plumb line and the ellipsoidal normal that intersect at point Crestone at the Earth's surface are 0.305 meters apart at the ellipsoidal surface. The ellipsoidal normal that marks the intersection of the plumb line with the ellipsoid has a longitude of W 105 degrees, 37 minutes, 29.98751 seconds. ($\Delta\lambda = \frac{\Delta easting}{N\cos\phi}$ where N is the radius of curvature of the ellipsoidal prime vertical for that point. $N = \frac{a}{\left(1 - e^2 \sin^2\phi\right)^{\frac{1}{2}}}$ where a, ϕ and e are given above. Thus $N = 6386244.475 meters$ and $\Delta\lambda = \frac{0.305}{N\cos(38^o)} = 0.01249\sec$) (Leick 1995))

6. EXAMPLES OF ELLIPSOIDAL SURFACE MODEL WITH FUZZY BOUNDARIES

Suppose a cadastre were to be developed for all the land located in Massachusetts. For this paper, a representative sample of DOV for various points in Massachusetts has been created (see Table 1) by scaling geodetic coordinates and orthometric heights from USGS topo maps and inputting these values into the NGS DEFLEC99 program to come up with the DOV.

One possible cadastre design would be to pick one DOV value that is as large as the largest DOV for any point in Massachusetts and pick one distance that governs the limits of ownership above and below the Earth's surface. With this design there would be one width value for the fuzzy boundary of every boundary line.

Another possible cadastre design would be to pick a DOV and a distance that governs the limits of ownership above and below the Earth's surface for each parcel or for a particular region (perhaps the size of a town or city). In this way the fuzzy boundary line width could be better fitted to each individual parcel or region.

6.1 Pick One Value of DOV for the Whole Cadastre

From Table 1, the largest DOV is 14.2 seconds for point Williamstown. For a 500-meter limit of ownership above and below the Earth's surface, every boundary line in the cadastre will have a fuzzy boundary line width of 0.069 meters.

6.2 Pick One Value of DOV for One Parcel or One Particular Region

For instance, the region around point Pocasset could be assigned the DOV of 6.2 seconds. For a 500-meter limit of ownership above and below the Earth's surface, each boundary line in the region will have a fuzzy boundary width of 0.030 meters.

7. CONCLUSION

In a future millimeter cadastre, the absolute location of a geodetic coordinate (latitude, longitude) may be repeatedly staked on the ground to one-millimeter accuracy. Since many jurisdictions allow a geodetic coordinate to definitively and legally mark a boundary corner, in the future a cadastre may be developed in which boundary corners of land ownership are definitively located on the Earth's surface not by physical monuments but by geodetic coordinates. Since ownership of land includes not only the soil and what is attached to it, but also the space beneath and above the surface, a cadastre will be made up of three-dimensional parcels of land ownership. A model of such a cadastre typically is viewed on a computer screen or paper map and thus must be a two-dimensional representation of the actual three-dimensional cadastre. This paper proposes such a two-dimensional model that is based on the ellipsoidal surface model in which the ellipsoidal normal is used to model the vertical dimension of land ownership through the use of the Deflection of the Vertical. The paper contains an extreme example for a point located in Colorado and then suggests two models for the creation of such a cadastre in Massachusetts. The error in the cadastral model created by using the ellipsoidal normal would be shown through the use of a "fuzzy" boundary line that would have a width sufficient to contain the said modeling error.

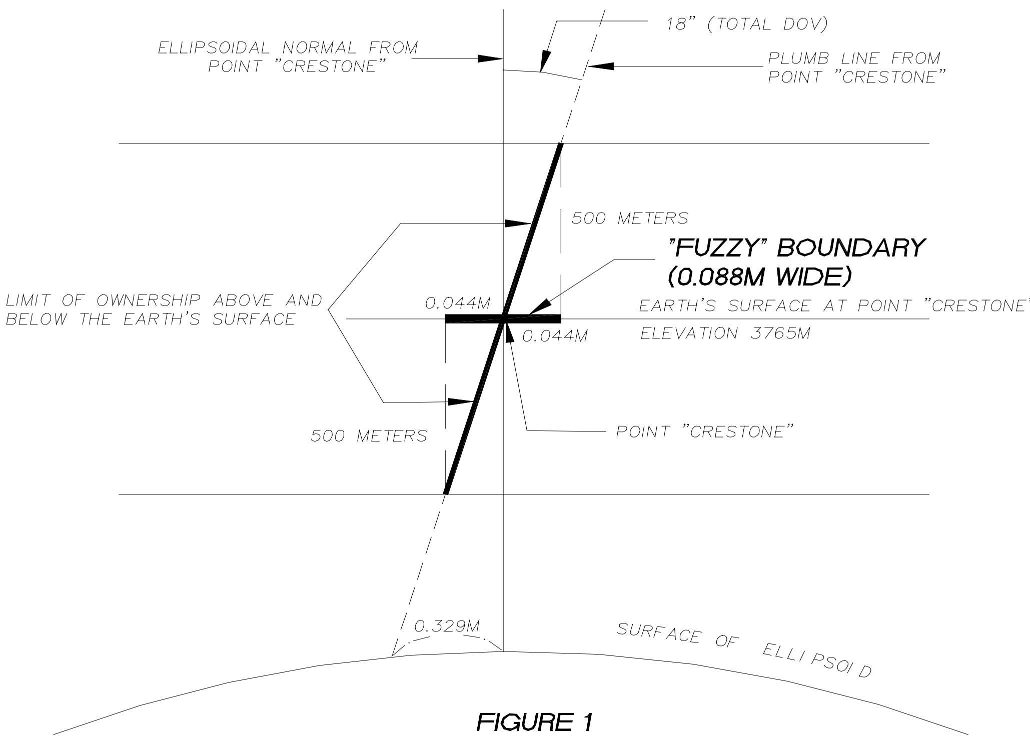

FIGURE 1

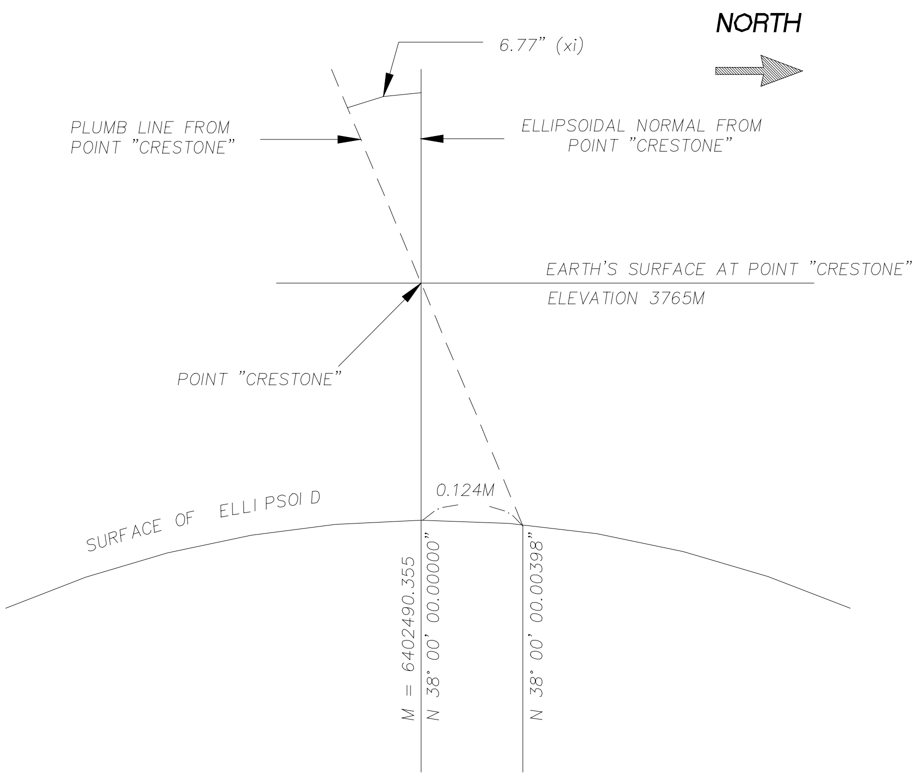

FIGURE 2

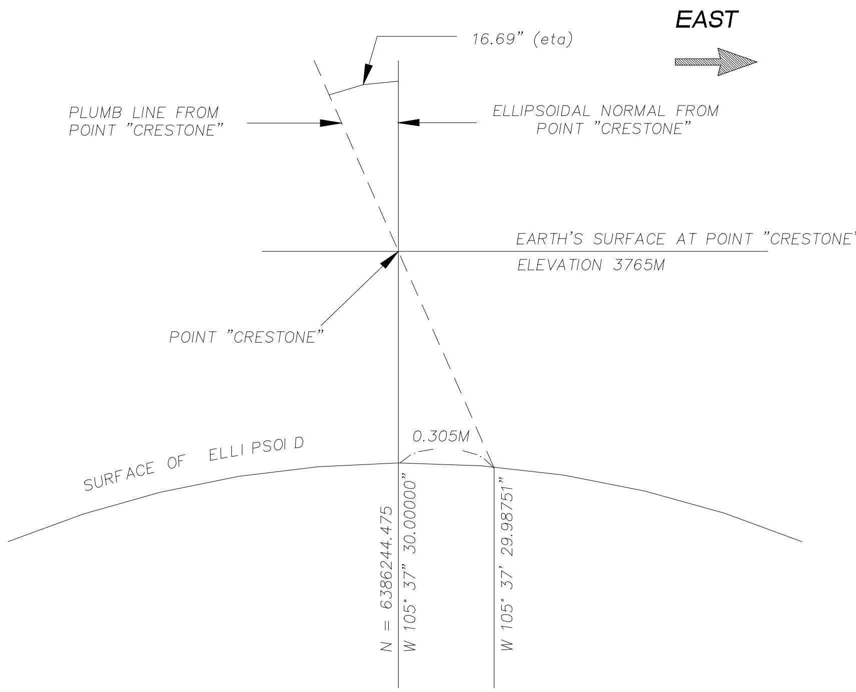

FIGURE 3

A	B	C	D	E	F	G	H	I
ashleyfalls	42.0340	-73.2900	645	-30.307	615	2.19	-12.34	12.5
cohasset	42.1445	-70.4800	15	-27.684	-13	-5.35	-2.53	5.9
greenfield	42.3730	-72.3720	60	-28.162	32	-3.21	1.65	3.6
newburyport	42.5000	-70.5500	15	-26.941	-12	-0.85	-3.83	3.9
mtholyoke	42.1920	-72.3420	64	-28.574	35	-2.45	-1.90	3.1
pittsfieldwest	42.2500	-73.2230	570	-30.260	540	1.10	-10.76	10.8
pocasset	41.4000	-70.3230	40	-28.367	12	-1.34	-6.01	6.2
springfieldsouth	42.0230	-72.3230	60	-28.962	31	-2.92	-0.31	2.9
townsend	42.4000	-71.3730	123	-27.779	95	-2.17	-0.94	2.4
webster	42.0500	-71.5000	222	-29.055	193	-4.51	1.27	4.7
williamstown	42.4000	-73.1230	290	-29.610	260	3.79	-13.70	14.2

Column A: Name of USGS Quad Sheet from which point was scaled
Column B: Latitude of scaled point (Degrees.MinutesSeconds)
Column C: Longitude of scaled point (Degrees.MinutesSeconds)
Column D: Orthometric height of scaled point (Meters)
Column E: Geoid undulation N for scaled point (Meters)
Column F: Ellipsoidal height for scaled point (Meters)
Column G: DOV along the meridian ξ (Seconds)
Column H: DOV along the prime vertical η (Seconds)
Column I: Total DOV (Seconds)

TABLE 1

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